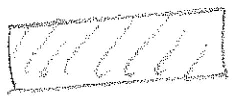
MATH 312
Lecture 16

DETERMINANTS.

Recap:
- Gram-Schmidt
- QR-decomposition.

$A = QR$
Q orthogonal
R upper triangular.

SUMMARY for $A\vec{x} = \vec{b}$

- If A looks like , i.e., more rows than columns, then it must have a null-space, so if $\vec{b}$ is in $C(A)$ then there are infinitely many solutions
- If A looks like  then probably $\vec{b}$ is NOT in $C(A)$. In this case, the best way forward is least-squares
- If A is square, anything can happen. Need more tools to understand square matrices.

DETERMINANT as FUNCTION.

There is ONLY one function

$$\underline{\det}: [n \times n \text{ matrices}] \longrightarrow [\text{Real numbers}]$$

which satisfies

- Normalization,
- Antisymmetry,
- Multilinearity.

Here,

N [• <u>Normalization</u> means $\det(\mathrm{id}) = 1$.

A [• <u>Antisymmetry</u> means $\det(B) = -\det(A)$ whenever B is obtained by swapping two rows.

• <u>Multilinearity</u> means

M_1 [1. If we scale a row of A by c to get B, then $\det(B) = c \cdot \det(A)$

M_2 [2. If we add a vector v to the k-th row of A to get B, then

$$\det(B) = \det(A) + \det(\text{blah}),$$

where blah is the matrix you'd get from replacing the k-th row of A by $\vec{v}$.

INTERPRETATION as VOLUME.

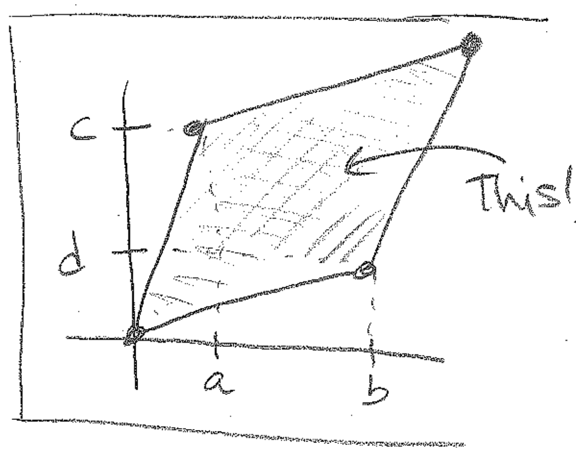
Determinants are VOLUMES (with a sign)

Let's stick to 2D, so we deal with <u>Areas</u>.

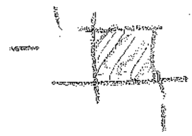
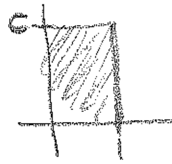
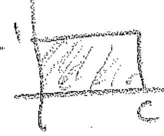
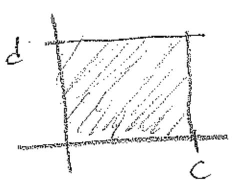
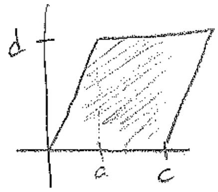
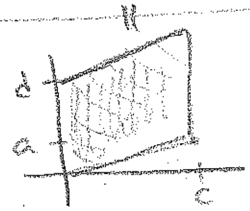
Given $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$,

$\det(A) = \pm$ Area of

to be determined...



Here is a table for 2×2 matrices:

MATRIX	VOLUME OF	USES	DET
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		N	1.
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$		N, A	-1
$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$, $\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$	 OR 	N, M_1	c.
$\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$		N, M_1	cd.
$\begin{bmatrix} c & a \\ 0 & d \end{bmatrix}$		N, M_1, M_2	cd.
$\begin{bmatrix} c & 0 \\ a & d \end{bmatrix}$		N, M_1, M_2	cd.

So: for triangular / diagonal matrices:

$$\det\begin{bmatrix} a_1 & \\ 0 & a_n \end{bmatrix} = \det\begin{bmatrix} a_1 & 0 \\ 0 & a_n \end{bmatrix} = \det\begin{bmatrix} a_1 & 0 \\ & a_n \end{bmatrix}$$

$= a_1 \cdot a_2 \cdot a_3 \cdots a_n$ = product of diagonal entries.

Row Operations:

Since we now know the determinants of triangular matrices, computing determinants of other matrices becomes easy via $A = LU$. But first, we need to know HOW row operations modify the determinant!

Type I: Add k times r_j to r_i

Type II: Swap r_i and r_j

Type III: Scale r_i by c.

By (A), a Type II operation scales the determinant by $\underline{-1}$.

By (M_1) a Type III operation scales the determinant by $\underline{c}$.

- But what about Type I operations? This is trickier.

Well, if $A = \begin{bmatrix} \text{---} \; r_1 \; \text{---} \\ \vdots \\ \text{---} \; r_i \; \text{---} \\ \vdots \\ \text{---} \; r_j \; \text{---} \\ \vdots \\ \text{---} \; r_n \; \text{---} \end{bmatrix}$

then, the Type 1 operation produces

$$\begin{bmatrix} — & r_1 & — \\ & \vdots & \\ — & r_i & — \\ & \vdots & \\ — & r_j + kr_i & — \\ & \vdots & \\ — & r_n & — \end{bmatrix}$$

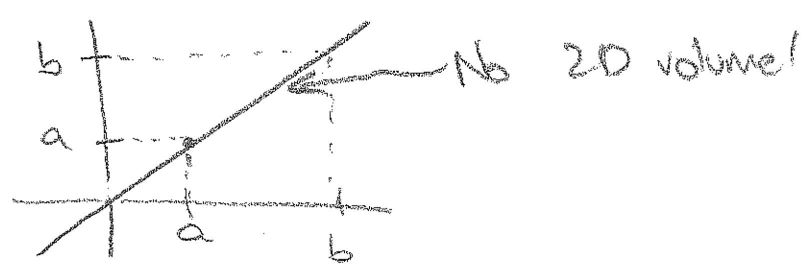
And by M_1, the det of this matrix is

$$\underbrace{\det \begin{bmatrix} — & r_1 & — \\ & \vdots & \\ — & r_i & — \\ & \vdots & \\ — & r_j & — \\ & \vdots & \\ — & r_n & — \end{bmatrix}}_{\text{This is just } \det A} + \underbrace{\det \begin{bmatrix} — & r_1 & — \\ & \vdots & \\ — & r_i & — \\ & \vdots & \\ — & kr_i & — \\ & \vdots & \\ — & r_n & — \end{bmatrix}}_{\text{And this is zero!}}$$

By M_2, we get $= k \det \begin{bmatrix} — & r_1 & — \\ & \vdots & \\ — & r_i & — \\ & \vdots & \\ — & kr_i & — \\ & \vdots & \\ — & r_n & — \end{bmatrix}$

<u>BUT</u> the volume of the parallelogram coming from a matrix with repeated rows is ZERO.

$$\begin{bmatrix} a & b \\ a & b \end{bmatrix} \rightsquigarrow$$



Similarly, in higher dimensions...

So, to compute $\det A$, where

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 5 \\ 1 & 2 & 1 \end{bmatrix}.$$

<u>OLD WAY</u>:

$$0\begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$$

No thanks!

<u>Better way</u>: Row-reduce to upper triangular and keep track of row ops!

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 5 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[r_1 \leftrightarrow r_2]{\overset{\text{change}}{-1}} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[r_3' = r_3 - r_1]{\text{No change}} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 0 & -1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 0 & -1 & -4 \end{bmatrix} \xrightarrow[r_3' = r_3 + r_2]{\text{no change}} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Now, $\det \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

= prod of diagonal entries

$= 1 \cdot 1 \cdot (-2) = -2.$

So, $\det(A) = \underbrace{-}_{\boxed{\text{from } r_1 \leftrightarrow r_2}}(-2) = \underline{\underline{2}}.$